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ABSTRACT

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A method is presented for solving a class of radiation and diffraction problems which have conventionally been formulated in terms of the Wiener-Hopf approach. The problem is formulated by representating the fields in terms of a modal expansion. The modal expansion for the field in the open regions is characterized by continuous eigenvalues and for the closed regions the fields are representated in terms of the discrete eigenvectors. Matching of the fields to the boundaries and across the aperture results in an equation which is solved by a technique developed in this article. The technique is an extension of the function-theoretic method necessitated by the field representation in the open regions inherent in the problems. The solutions of an open-ended parallel plate wavequide is used to demonstrate the method. Applications to other geometries are indicated.

1 Introduction

The object of this paper is to discuss a technique for solving a class of radiation and diffraction problems which have conventionally been formulated in terms of the Wiener-Hopf technique.* In essence, the present method is an extension of the function-theoretic technique introduced by Whitehead [1951] for solving an infinite set of equations associated with the problem of diffraction by an infinite stack of periodically-spaced half-planes. Numerous other authors (see for instance, Hurd and Gruenberg [1954], Mittra [1959], Pace and Mittra [1964] and Karjala and Mittra [1965]) have demonstrated the usefulness of the method of formulation in terms of an infinite set of equations through applications to a wide variety of problems. The geometries associated with the above problems have the common feature that in general they are related to, and are modifications of, certain basic configurations which may be classified as the Wiener-Hopf type. However, to date the application of the function-theoretic technique has been confined either to the closed-region problems such as discontinuities in a waveguide, or to certain open-region problems, which because of their periodic nature, may be related to some equivalent closed-region problem. Also, previous workers using the function-theoretic approach have restricted themselves to geometries conforming to the cartesian system only. In contrast, it is well known that the Wiener-Hopf technique is applicable to some semi-infinite geometries in the cylindrical system and to a number of open-region problems.

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* For exhaustive discussion and numerous illustrations of the Wiener-Hopf technique see Noble [1958].

In this paper we extend the function-theoretic approach to some aperiodic open-region geometries. To illustrate the technique, the problem of an open-ended parallel plane waveguide is considered in some detail. Applications to other geometries, such as the open-ended circular waveguide or the radiation from an open-ended inhomogeneous parallel plane waveguide are indicated. Some related problems which may be regarded as modifications of the basic Wiener-Hopf geometry are discussed briefly.

2 Radiation from a Parallel Plate Waveguide

2.1 Formulation of the problem

Let two parallel plates at $x = -b, b, z < 0$ form an open-ended waveguide as shown in Fig. 1.

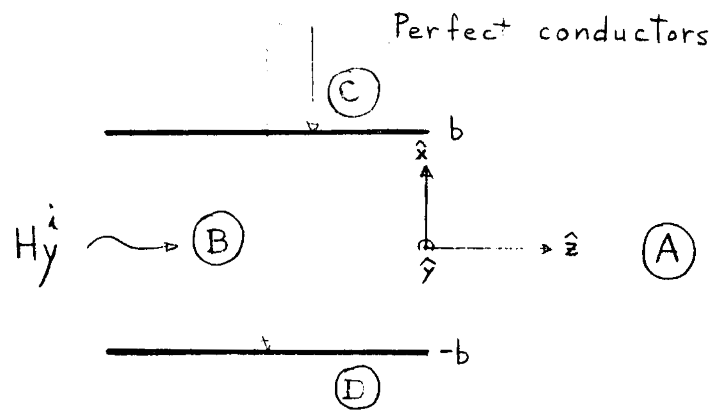

Fig. 1. Open-ended parallel plate waveguide

The incident field is assumed to be the TEM mode with the magnetic field intensity vector parallel to the walls of the guide. Since the incident field is independent of the y-coordinate and the entire structure is uniform with respect to the y-axis, the total field will also be independent of y. Therefore we wish to solve the two-dimensional wave equation for the scalar potential Φ

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial^2 z} + k_0^2\phi = 0 \tag{2.1}$$

All of the field quantities are derivable from Φ by letting $H_y = \Phi$, $E_x = \frac{1}{j\omega\epsilon}\frac{\partial\Phi(*)}{\partial z}$ and $E_z = \frac{-1}{j\omega\epsilon}\frac{\partial\Phi}{\partial x}$. The following boundary conditions apply on Φ:

a) Φ and $\nabla\Phi$ are finite in all regions, except at the edge of the guide where $|\nabla\Phi|$ becomes infinite according to the edge condition [Meixner, 1954] as $r^{-\frac{1}{2}}$.

b) Φ and $\partial\Phi/\partial x$ are continuous across the interface at z = 0.

c) $\partial\Phi/\partial x$ vanishes on the walls of the guide at x = b and x = -b.

d) Apart from the incident component, Φ satisfies the appropriate radiation condition for large distances away from the origin.

e) Φ satisfies the edge condition as the edge of the plate is approached, that is, goes to zero as $r^{\frac{1}{2}}$.

The space in which the guide is immersed is divided into four regions for convenience and are designated as regions A, B, C, D as seen in Fig. 1. The expressions for Φ which satisfy (2.1) and are valid for each region may be written as

$$\phi_B = Be^{-\beta_0 z} + \sum_{0}^{\infty} R_m \cos\frac{m\pi}{b}(x+b)\, e^{\beta_m z} \tag{2.2}$$

$$\phi_A = \int_{C_1} A(\alpha)\cos(\alpha x)\, e^{-\tau z}\, d\alpha \tag{2.3}$$

* $e^{-j\omega t}$ time convention.

$$\phi_C = \int_{C_2} C(\gamma) \cos\gamma(x-b)\, e^{\eta z}\, d\gamma \tag{2.4}$$

$$\phi_D = \int_{C_2} C(\gamma) \cos\gamma(-x-b)\, e^{\eta z}\, d\gamma \tag{2.5}$$

where B is the magnitude of the incident H field. C_1 and C_2 are contours going from zero to infinity and passing below the branch point at k_0, if k_0 is real. Also

$$\beta_m = \sqrt{\left(\frac{m\pi}{b}\right)^2 - k_0^2} = -j\sqrt{k_0^2 - \left(\frac{m\pi}{b}\right)^2} \tag{2.6}$$

$$\eta = \sqrt{\gamma^2 - k_0^2} = -j\sqrt{k_0^2 - \gamma^2} \tag{2.7}$$

$$\tau = \sqrt{\alpha^2 - k_0^2} = -j\sqrt{k_0^2 - \alpha^2} \tag{2.8}$$

$$k_0 = \omega\sqrt{\mu_0 \epsilon_0} \tag{2.9}$$

The branches of β_m, η, τ were chosen to give outgoing waves. That is, if k_0 is complex, then $k_0 = k_1 + jk_2$, where k_1, $k_2 > 0$. Also the real part of β_m is greater than zero and the real parts of η and τ are greater than zero along the contours of integration if the branches are chosen so that (2.6), (2.7) and (2.8) hold. Obviously conditions (c) and (d) are satisfied by the form of equations (2.2) to (2.5) and definitions (2.6), (2.7) and (2.8). The symmetric form used in equations (2.2) to (2.5) was dictated by the fact that both the incident Φ and the geometry are symmetric about $x = 0$.

Continuity of H_y and E_x at the interface requires that the following equations hold at $z = 0$:

$$\int_{c_2} C(\gamma)\cos\gamma(x-b)\,d\gamma = \int_{c_1} A(\alpha)\cos\alpha x\,d\alpha \quad ; \quad b \leq x \tag{2.10}$$

$$B + \sum_{m=0}^{\infty} B_m \cos\frac{m\pi}{b}(x+b) = \int_{c_1} A(\alpha)\cos\alpha x\,d\alpha \quad ; \quad -b \leq x \leq b \tag{2.11}$$

$$\int_{c_2} C(\gamma)\cos\gamma(-x-b)\,d\gamma = \int_{c_1} A(\alpha)\cos\alpha x\,d\alpha \quad ; \quad x \leq -b \tag{2.12}$$

and

$$\int_{c_2} \eta C(\gamma)\cos\gamma(x-b)\,d\gamma = -\int_{c_1} \tau A(\alpha)\cos\alpha x\,d\alpha \quad ; \quad b \leq x \tag{2.13}$$

$$-\beta_0 B + \sum_{m=0}^{\infty} \beta_m B_m \cos\frac{m\pi}{b}(x+b) = -\int_{c_1} \tau A(\alpha)\cos\alpha x\,d\alpha \quad ; \quad -b \leq x \leq b \tag{2.14}$$

$$\int_{c_2} \eta C(\gamma)\cos\gamma(-x-b)\,d\gamma = -\int_{c_1} \tau A(\alpha)\cos\alpha x\,d\alpha \quad ; \quad x \leq -b \tag{2.15}$$

The solutions of equations (2.10) to (2.15) for the unknown functions $C(\gamma)$, $A(\alpha)$ and the infinite number of unknown coefficients B_m will yield a solution for the fields by insertion into equations (2.2) to (2.5). The solution of these unknowns is discussed next.

2.2 Solution of B_m, $A(\alpha)$ and $C(\gamma)$

In order to solve for the unknowns, first eliminate the x variation from (2.10) through (2.15) by multiplying by $\sqrt{2\pi}\,e^{i\alpha x}$ and integrating from

$-\infty$ to ∞. This gives

$$\frac{2B\alpha\sin(\alpha b)}{\sqrt{2\pi}\,(\tau^2-\beta_0^2)} + \sum_{m=0}^{\infty}\frac{B_m\,2\alpha\sin(\alpha b)}{\sqrt{2\pi}\,(\tau^2-\beta_m^2)}$$

$$-\frac{2\alpha\sin(\alpha b)}{\sqrt{2\pi}}\int_{C_2}\frac{C(\gamma)}{\tau^2-\eta^2}\,d\gamma = \sqrt{\frac{\pi}{2}}\,A(\alpha) \tag{2.16}$$

$$\frac{-2B\beta_0\,\alpha\sin(\alpha b)}{\sqrt{2\pi}\,(\tau^2-\beta_0^2)} + \sum_{m=0}^{\infty}\frac{B_m\,\beta_m\,2\alpha\sin(\alpha b)}{\sqrt{2\pi}\,(\tau^2-\beta_m^2)}$$

$$-\frac{2\alpha\sin(\alpha b)}{\sqrt{2\pi}}\int_{C_2}\frac{\eta\,C(\gamma)}{\tau^2-\eta^2}\,d\gamma = -\tau\sqrt{\frac{\pi}{2}}\,A(\alpha) \tag{2.17}$$

The B_m coefficients may be obtained by first eliminating $A(\alpha)$ from (2.16) and (2.17). This is done by multiplying (2.16) by τ and adding to (2.17) to get

$$\frac{B}{\tau+\beta_0} + \sum_{m=0}^{\infty}\frac{B_m}{\tau-\beta_m} - \int_{C_2}\frac{C(\gamma)}{\tau-\eta}\,d\gamma = 0 \tag{2.18}$$

It is interesting to observe that for the closed structure, that is, one in which the waveguide is enclosed within a larger waveguide as indicated in Figure 2, the corresponding equation is

$$-\sum_{m=0}^{\infty}\frac{C_m}{\tau_p-\eta_m} + \sum_{m=0}^{\infty}\frac{B_m}{\tau_p-\beta_m} + \frac{B}{\tau_p+\beta_0} = 0\ ; \qquad p = 0, 1, 2, \cdots \tag{2.19}$$

The above equation may be obtained by expanding Φ in the four regions A, B, C, and D, matching Φ and $\partial\Phi/\partial z$ at $z = 0$ and eliminating the A_m's. Here:

C_m are the coefficients of the normal modes in regions C and D.

A_m are the coefficients of the normal modes in region A.

$$\eta_m = \sqrt{\left(\frac{m\pi}{c}\right)^2 - k_0^2} = -j\sqrt{k_0^2 - \left(\frac{m\pi}{c}\right)^2} \tag{2.20}$$

$$\tau_p = \sqrt{\left(\frac{p\pi}{a}\right)^2 - k_0^2} = -j\sqrt{k_0^2 - \left(\frac{p\pi}{a}\right)^2} \tag{2.21}$$

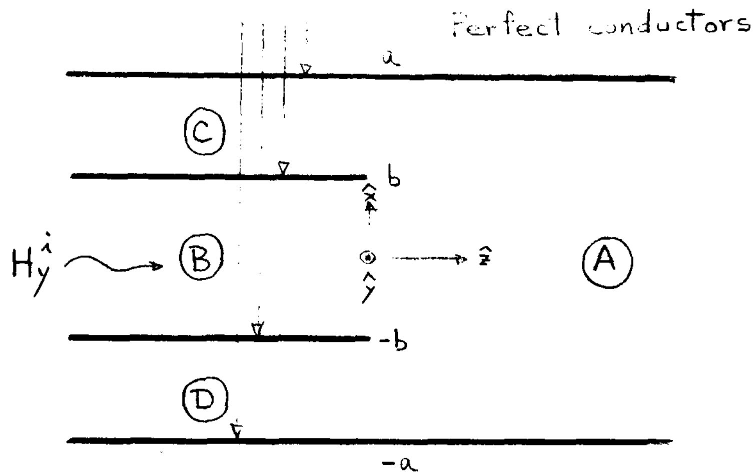

Fig. 2. Coupled parallel plate waveguides

Equation (2.18) may be regarded as a limiting case of (2.19) in the following sense: as a, c approach infinity, such that a - c = b is maintained, the propagation constants η_m and τ_p approach a continuous spectrum and the sum of the normal modes becomes an integral over a contour which may be associated with the continuous version of the eigenvalue spectrum. Thus (2.19) approaches (2.18) as a limiting case.

To extract the B_m coefficient from equation (2.18), we shall employ an extension of the function-theoretic method [Hurd and Gruenberg, op. cit.] used in the coupled waveguide problem. We select a function H(w) which has branch points at β_0 and infinity, is analytic elsewhere except at simple poles at $w = \beta_m$ and $w = -\beta_0$, and goes to zero at least as fast as $|w|^{-\nu}$, $|w| \to \infty$, $\nu > 0$. This function may be obtained by reference to the coupled waveguide problem as explained in section 3.

Consider the following integral

$$\frac{1}{2\pi j}\int_S \frac{H(\omega)}{\tau-\omega}\,d\omega \tag{2.22}$$

where: $$H(\omega) = \frac{C_o\,R(\omega)}{\Pi(\omega,\beta)(\omega+\beta_o)(\omega-\beta_o)} \tag{2.23}$$

$H(w) \rightarrow |w|^{-3/2}$ as $|w| \rightarrow \infty$ in the upper half plane and decays exponentially in the lower half plane.

C_o = unknown polynomial in w but is equal to a constant in this case to satisfy the edge condition, (refer to section 3), and

$$\Pi(\omega,\beta) = \prod_{m=1}^{\infty}\left(1-\frac{\omega}{\beta_m}\right)e^{\frac{\omega}{\beta_m}} \tag{2.24}$$

$$R(\omega) = \mathrm{Exp}\left\{-\frac{b\omega}{\pi}\left[1-C_e+\ln\left(\frac{2\pi}{k_o b}\right)\right] - j\frac{b\omega}{2} - \frac{b}{\pi}\sqrt{k_o^2+\omega^2}\,\ln\left(\frac{\omega-\sqrt{k_o^2+\omega^2}}{j k_o}\right)\right\} \tag{2.25}$$

S = closed contour chosen to enclose the poles at β_m, m = 1,2,..., and $-\beta_0$, to exclude the branch cut and the point τ, and to close in an infinite circle.

The contour S is shown in Fig. 3.

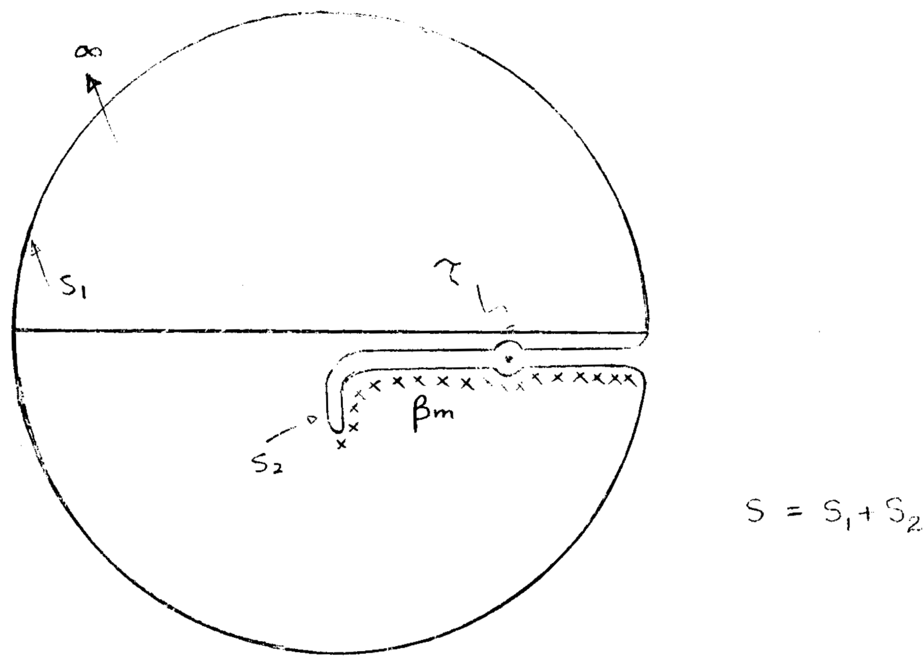

Fig. 3. Contour of H(w) integration in equation (2.26).

Using the above properties of H(w) in (2.22), there is obtained

$$\frac{1}{2\pi j}\int_{S_1}\frac{H(\omega)}{\tau-\omega}\,d\omega + \frac{1}{2\pi j}\int_{S_2}\frac{H(\omega)}{\tau-\omega}\,d\omega \tag{2.26}$$
$$= \Sigma \text{ Residues at } \beta m, -\beta_o$$

The contour S_1 is the infinite circle and hence the first term of (2.26) is zero because of the asymptotic nature of H(w). The contour S_2 is the contour around the branch cut and indented around the point τ. From (2.26) one derives

$$\sum_{m=o}^{\infty}\frac{r(\beta m)}{\tau-\beta m} + \frac{r(-\beta_o)}{\tau+\beta_o}$$
$$-\frac{1}{2\pi j}\int_{S2}\frac{H(\omega)}{\tau-\omega}\,d\omega = 0 \tag{2.27}$$

where $r(\beta_m)$ is the residue of $H(w)$ evaluated at $w = \beta_m$.

A comparison of (2.27) with (2.18) suggests that we set

$$B = r\,(-\beta_o) \tag{2.28}$$

$$B_m = r(\beta_m)\ ;\ m = 0,\ 1,\ 2, \ldots \tag{2.29}$$

Then

$$\frac{1}{2\pi j}\int_{S_2}\frac{H(\omega)}{\tau-\omega}\,d\omega = \int_{C_2}\frac{C(r)}{\tau-\eta}\,dr \quad , \quad \text{for all } \tau \tag{2.30}$$

The unknown constant C_o in $H(w)$ is obtained from (2.28) and gives

$$C_o = \frac{-2\beta_o B\,\Pi(-\beta_o,\beta)}{R(-\beta_o)} \tag{2.31}$$

Substituting (2.31) in (2.23) and evaluating (2.29) gives

$$B_o = -B\,\frac{\Pi(-\beta_o,\beta)\;R(\beta_o)}{\Pi(\beta_o,\beta)\;R(-\beta_o)} \tag{2.32}$$

and

$$B_m = \frac{2\beta_o B b\, e^{-\frac{b\beta m}{m\pi}}\;R(\beta_m)}{m\pi\;R(-\beta_o)\;\Pi^{(m)}(\beta_m,\beta)}\ ;\ m = 1, 2, \cdots \tag{2.33}$$

where $\Pi^{(m)}(\beta_m,\beta)$ is the product as defined in (2.24) but with the m-th term deleted.

The function $A(\alpha)$ is obtained from (2.15) and (2.16) by multiplying (2.15) by τ and subtracting (2.16) to obtain

$$\frac{B}{\tau-\beta_o} + \sum_{m=0}^{\infty}\frac{B_m}{\tau+\beta_m} - \int_0^{\infty}\frac{C(r)}{\tau+\eta}\,dr$$

$$= \frac{\pi\tau A(\alpha)}{\alpha\sin(\alpha b)} = A'(\alpha) \tag{2.34}$$

Again integrating $H(w)$ around a suitable contour in the w-plane which incloses the points β_m, $-\beta_0$, and this time $-\tau$, we have

$$\frac{1}{2\pi j}\int_{S_1+S_2}\frac{H(\omega)}{\tau+\omega}\,d\omega = \sum_{m=0}^{\infty}\frac{r(\beta_m)}{\tau+\beta_m} + \frac{r(-\beta_0)}{\tau-\beta_0} + H(-\tau) \tag{2.35}$$

Since $\int_{S_1}$ is again zero, we are left with

$$\frac{r(-\beta_0)}{\tau-\beta_0} + \sum_{m=0}^{\infty}\frac{r(\beta_m)}{\tau+\beta_m} - \frac{1}{2\pi j}\int_{S_2}\frac{H(\omega)}{\tau+\omega}\,d\omega = -H(-\tau) \tag{2.36}$$

A comparison of (2.36) with (2.34) and recalling (2.28), (2.29), and (2.30) gives

$$A'(\alpha) = -H(-\tau) = -H\left(-\sqrt{\alpha^2-k_0^2}\right) \tag{2.37}$$

So

$$A(\alpha) = \frac{2}{\pi}\beta_0 B\,\frac{\sin(\alpha b)}{\alpha\sqrt{\alpha^2-k_0^2}}\;\frac{\Pi(-\beta_0,\beta)}{\Pi(-\sqrt{\alpha^2-k_0^2},\beta)}\;\frac{R(-\sqrt{\alpha^2-k_0^2})}{R(-\beta_0)} \tag{2.38}$$

The remaining unknown quantity, viz. $C(\gamma)$ is obtained by reference to (2.30) which relates $C(\gamma)$ to $H(w)$. As a first step one transforms the path of integration in the w plane so as to make it identical to the path C_2. This is achieved in two steps as follows. First rewrite the $H(w)$ integral as

$$\frac{1}{2\pi j}\int_{S_2}\frac{H(\omega)}{\tau-\omega} = \frac{1}{2\pi j}\int_{S_2^+}\frac{H(\omega)}{\tau-\omega}\,d\omega - \frac{1}{2\pi j}\int_{S_2^-}\frac{H(\omega)}{\tau-\omega}\,d\omega \tag{2.39}$$

where the paths S_2^+ and S_2^- refer to the upper and lower portions of the branch cut integral S_2. Using the relationship between the values of $H(w)$ on the two paths, viz., $H(\omega)\big|_{S_2^-} = e^{jb\sqrt{\omega^2+k^2}}\,H(\omega)\big|_{S_2^+}$, which is derivable from the known expression of H(w), one may obtain

$$\frac{1}{2\pi j}\int_{S_2}\frac{H(\omega)}{\tau-\omega}\,d\omega = -\frac{1}{\pi}\int_{S_2^+}\frac{H(\omega)}{\tau-\omega}\,e^{jb\sqrt{k_0^2+\omega^2}}\sin\left(b\sqrt{k_0^2+\omega^2}\right)d\omega \tag{2.40}$$

Now introduce a change in the variable of integration in the right hand side of (2.40), according to

$$\sqrt{r^2-k_0^2} = \omega \tag{2.41}$$

This yields the desirable form

$$\frac{1}{2\pi j}\int_{S_2}\frac{H(\omega)}{\tau-\omega}\,d\omega = -\frac{1}{\pi}\int_{C_2} e^{jbr}\,\frac{H(\eta)\sin(br)}{\eta}\,r\,dr \tag{2.42}$$

where η is given by (2.7).

Comparing (2.30) and (2.42) one obtains

$$C(r) = \frac{-1}{\pi}\,e^{jbr}\,H(\eta)\,\frac{\sin(br)\,r}{\eta} \tag{2.43}$$

All the unknowns are now determined and the solution of the problem is complete.

Before leaving this discussion we shall mention an alternative formulation in which one eliminates the unknowns B_m and $C(\gamma)$, retaining $A(\alpha)$ in the resulting equations. The elimination procedure is very similar to the one employed by Hurd [op. cit.], the only difference being that the transforms rather than series representations for Φ and its derivative are matched at the interface of regions A and C (or D).

The coupled integral equations for $A_1(\alpha) = A(\alpha)\alpha\sin nb\alpha$ are given by

$$\int_C \frac{A_1(\alpha)}{\tau-\eta}\,d\alpha = 0$$

$$\int_c \frac{A_1(\alpha)}{\tau - \beta_m} = 2\,\delta_m^\circ B b \beta_\circ \; ; \qquad \delta_m^\circ = \begin{cases} 1, & m=0 \\ 0, & m \neq 0 \end{cases} \tag{2.45}$$

Two related equations giving B_m's and $C(\gamma)$ in terms of $A(\alpha)$ are

$$\int_c \frac{A_1(\alpha)}{\tau + \eta}\, d\alpha = \pi C(\gamma)\eta \tag{2.46}$$

$$-\int_c \frac{A_1(\alpha)}{\tau + \beta_m}\, d\alpha = 2\beta_\circ B_\circ b\, \delta_m^\circ + B_m \beta_m b\,(1 - \delta_m^\circ) \tag{2.47}$$

The solution of the (2.44) and (2.45) may be accomplished by constructing a function H'(w) which is given by

$$H'(\omega) = \frac{C'\,\pi(\omega, \beta)}{R(\omega,} \tag{2.48}$$

where the quantities appearing in (2.48) are the same as defined before except for the constant C' which is given by

$$C' = \frac{2Bb\beta_\circ R(\beta_\circ)}{\pi(\beta_\circ, \beta)} \tag{2.49}$$

The resulting expressions for the unknowns $C(\gamma)$ and B_m's, derived from the use of function-theoretic technique involving a complex integration in the w-plane are given by

$$B_m = -\frac{H'(-\beta_m)}{\beta_m b}, \qquad B_\circ = -\frac{H'(-\beta_\circ)}{2b\beta_\circ} \tag{2.50}$$

$$C(\gamma) = \frac{H'(-\eta)}{\pi\eta} \tag{2.51}$$

The function $A(\alpha)$ may also be expressed in terms of $H'(w)$ in the same manner $C(\gamma)$ was related to $H(w)$ in the previous discussion.

3 Construction of H(w)

In the previous section the solution of (2.18) was based on being able to obtain a function $H(w)$ satisfying certain specified conditions which were set forth in Section 2. In the following we discuss a method for the derivation of $H(w)$.

The insight for constructing $H(w)$ comes from the procedure used in the closed waveguide problem (Fig. 2). To solve the closed guide equation, i.e. (2.19), a function is chosen to have:

a) poles at β_m, $m = 1,2,\ldots$, β_o, $-\beta_o$, η_m, $m = 1,2,\ldots$

b) zeros at τ_p, $p = 1, 2,\ldots$

c) asymptotic behavior as $|w|^{-\nu}$, $\nu > 0$, as $|w| \to \infty$

As in the open case, ν depended on the edge condition. For the closed case $H_c(w)$ is taken to be

$$H_c(\omega) = \frac{\Pi(\omega,\tau)\ g_c(\omega)}{\Pi(\omega,\eta)\ \Pi(\omega,\beta)\ (\omega+\beta_o)(\omega-\beta_o)} \tag{3.1}$$

where $\Pi(w,\tau)$ and $\Pi(w,\eta)$ are obtained by replacing b by a and c, respectively, in the expression for $\Pi(w,\beta)$ (see 2.24), and $g_c(w)$ is chosen to give proper asymptotic behavior of $H_c(w)$.

Equation (3.1) admits an alternate representation

$$H_c(\omega) = \frac{g_c(\omega)}{\Pi(\omega,\beta)(\omega^2-\beta_o^2)} R_c(\omega) \tag{3.2}$$

$$R_c(\omega) = \exp\left\{ \int_\Sigma \left[\ln\left(1-\frac{\omega}{\sqrt{\gamma^2-k_o^2}}\right) + \frac{\omega}{\sqrt{\gamma^2-k_o^2}} \right] \left(\frac{F_1'(\gamma)}{F_1(\gamma)} - \frac{F_2'(\gamma)}{F_2(\gamma)} \right) d\gamma \right\} \tag{3.3}$$

where:

a) $F_1(\gamma)$ has simple zeros at $\frac{n\pi}{a}$, n = 1, 2,...

b) $F_2(\gamma)$ has simple zeros at $\frac{n\pi}{c}$, n = 1, 2,...

c) $F_1' = dF_1/d\gamma$, $F_2' = dF_2/d\gamma$

d) Σ is the contour shown in Fig. 4.

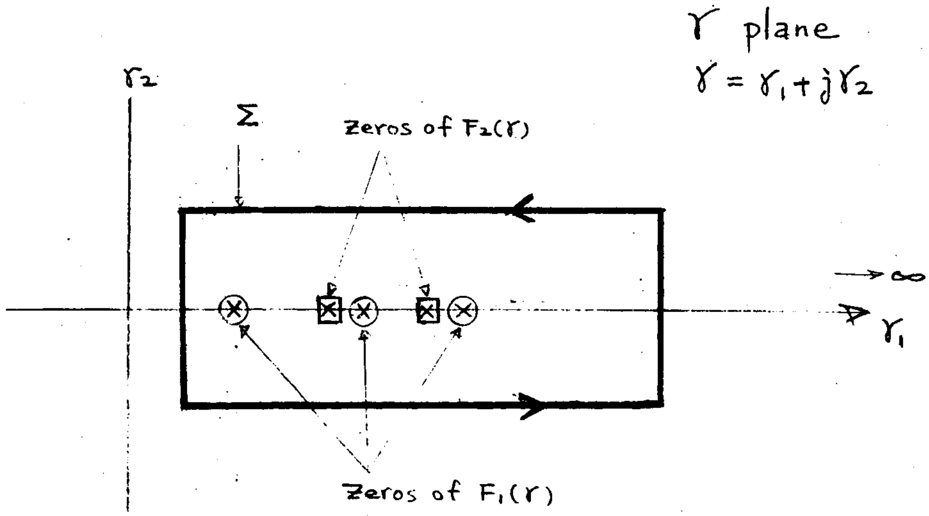

Fig. 4. Contour used in the integral representation of $R_c(w)$

$F_1(\gamma)$ and $F_2(\gamma)$ may be identified with the functions which when set to zero, result in the characteristic equations for the regions A and C (or D), respectively (see Fig. 2). In this particular problem

$$F_1(\gamma) = \sin \gamma a \tag{3.4}$$

$$F_2(\gamma) = \sin \gamma c \tag{3.5}$$

As we shall see shortly, the particular form of representation in (3.3) is useful in deriving H(w) by taking the limit of $R_c(w)$ as $a \rightarrow \infty$, $c \rightarrow \infty$, $a - c = b$. Another important remark in connection with (3.3) concerns its generality, in that it applies to several other geometries involving the junction of a semi-infinite waveguide to another infinite guide of larger dimensions. It is found that in these cases the form of (3.3) remains unchanged. One needs only to substitute the characteristic expressions for $F_1(\gamma)$ and $F_2(\gamma)$, appropriate for the particular geometry under consideration. The last point is discussed further in Section 4.

Returning to the problem at hand, we recall that in Section 2, it was discussed in connection with (2.19) that the discrete propagation constants η_m and τ_p approach a continuum going from β_o to infinity as a, $c \rightarrow \infty$, such that $a - c = b$ is maintained. This suggests that in the open case we choose

$$H(\omega) = \frac{\mathcal{G}(\omega)\, R_1(\omega)}{\Pi(\omega,\beta)\,(\omega^2 - \beta_o^2)} \tag{3.6}$$

where

$$R_1(\omega) = \lim_{\substack{a \to \infty \\ c \to \infty}} R_c(\omega) \tag{3.7}$$

and g(w) is determined by satisfying the requirement on the asymptotic behavior of H(w). Taking the limits of the ratios F_1'/F_1 and F_2'/F_2 along the upper and lower portions of the Σ path in the complex plane and rearranging the results, one obtains from (3.3) and (3.7):

$$R_1(\omega) = \exp\left\{\frac{b}{\pi}\int_0^\infty \left[\ln\left(1-\frac{\omega}{\sqrt{\gamma^2-k_0^2}}\right)+\frac{\omega}{\sqrt{\gamma^2-k_0^2}}\right]d\gamma\right\} \tag{3.8}$$

This may be integrated to give

$$R_1(\omega) = \exp\left\{-\frac{b}{\pi}\sqrt{k_0^2+\omega^2}\,\ln\left(\frac{\omega-\sqrt{k_0^2+\omega^2}}{jk_0}\right)+j\frac{k_0 b}{2}-\frac{b\omega}{\pi}\right\} \tag{3.9}$$

and the branch of $\sqrt{(k_o^2 + w^2)}$ is chosen to give the branch point of $R_1(w)$ at $w = \beta_o = -jk_o$.

The function g(w) in (3.7) is employed to give the proper asymptotic form of H(w). To insure that the function Φ behaves as $r^{1/2}$ as the edge of the guide is approached requires that H(w) have algebraic behavior, namely $w^{-3/2}$, at the points $-\beta_m$ or along the curve traced out by $-\tau$ as α varies over the contour C_1 in eq. (2.3), Section 2. This asymptotic behavior is deduced by studying either the behavior of the reflection coefficient B_m [Hurd, op. cit.] for large m or the behavior of $A(\alpha)$ [Noble, op. cit.] for large α. One can verify that the edge condition requires that choice for g(w) to be

$$g(\omega) = C_0 e^{-\frac{b\omega}{\pi}\left(-C_e+\ln\frac{2\pi}{k_0 b}\right)-j\frac{b\omega}{2}} \tag{3.10}$$

where:

C_e = Euler's constant = 0.5772...

C_o = unknown constant

Therefore

$$H(\omega) = \frac{C_o R(\omega)}{\Pi(\omega, \beta)\,(\omega+\beta_o)(\omega-\beta_o)} \tag{3.11}$$

where:

$$R(\omega) = \exp\left\{-\frac{b\omega}{\pi}\left[1 - C_e + \ln\left(\frac{2\pi}{k_o b}\right)\right] - j\frac{b\omega}{2} - \frac{b}{\pi}\sqrt{k_o^2+\omega^2}\,\ln\left(\frac{\omega - \sqrt{k_o^2+\omega^2}}{j\,k_o}\right)\right\} \tag{3.12}$$

The constant part of $R_1(w)$, namely $\exp(jk_o b/2)$ has been included in C_o.

It may be verified that the function $H(w)$ given in eq. (3.11) has asymptotic behavior of $|w|^{-3/2}$ in the upper half plane and decays exponentially in the lower half plane for large w. At any rate, it satisfies the conditions that it goes to zero for large $|w|$ at least as $|w|^{-3/2}$ and further that the coefficients β_m and the function $A(\alpha)$ derived from it conform to the asymptotic behavior required by the edge condition.

This completes the discussion of the procedure for construction of the function $H(w)$.

4 Application to other geometries

We shall briefly discuss certain other configurations to which the procedure illustrated above in the case of an open-ended waveguide problem is applicable. The key to the general approach is the recognition of the fact that the solution of the open region problem is derivable as a limit of the closed region problem using an appropriate limiting procedure. In the function-theoretic technique the limiting procedure is applied to the function $H_c(w)$. The residues of $H(w)$ evaluated at the singularities of this function are identified with the unknown mode coefficients in the appropriate regions (regions B and C for instance in the coupled waveguide problem shown in Fig. 2), and that $H(w)$ itself evaluated at certain other points in the w-plane yields the mode coefficients for the remaining region, viz., region A in Fig. 2.

There are numerous closed waveguide problems which are solvable by an application of the function-theoretic technique. Consider for instance a semi-infinite cylindrical waveguide opening into a larger cylindrical waveguide, both having a common axis. The geometry of the problem is shown in Fig. 5a.

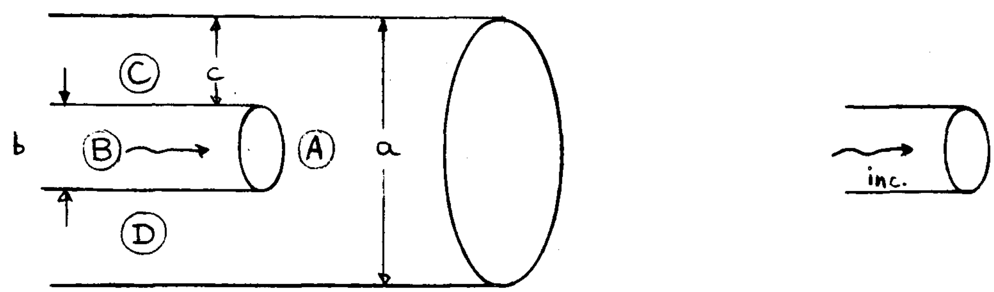

Fig. 5. Cylindrical waveguide problems to which the function-theoretic technique is applicable

The problem associated with this geometry is solvable by an application of a procedure identical to the one used for the parallel plane waveguide configuration. Expanding the fields in the various regions in terms of the appropriate mode functions, and applying the continuity of the fields, it is possible to obtain an equation having a form exactly similar to (2.19). Thus the form of $H_c(w)$ given in (3.2) remains unchanged and consequently $R_c(w)$ appearing in (3.3) also remains unchanged except for the fact that the functions $F_1(\gamma)$ and $F_2(\gamma)$ now correspond to the characteristic functions associated with the regions A and C, respectively, of the cylindrical and coaxial waveguides of Fig. 5a. Thus one has

$$F_1(\gamma) = J_0(\gamma a) \tag{4.1}$$

and

$$F_2(\gamma) = J_0(\gamma a) N_0(\gamma b) - J_0(\gamma b) N_0(\gamma a) \tag{4.2}$$

Recall that the zeros of $F_1(\gamma)$ and $F_2(\gamma)$ are the eigenvalues corresponding to the regions A and C respectively. It is assumed in writing (4.1) and (4.2) that the scalar potential Φ is circularly symmetric and that it satisfies the Dirichlet condition on the walls. However, the presence of an azimuthal variation or the introduction of Neumann or mixed type boundary conditions present no additional difficulties.

To construct the function H(w) appropriate for the open-ended cylindrical waveguide problem shown in Fig. 5b, one again makes use of the limiting procedure discussed in Section 3. Substituting (4.1) and (4.2) in the integral representation (3.3) for $R_c(w)$, one approaches the limit $a, c \to \infty$, with $a - c = b$. The resulting expression for R(w) in this case is

$$R(\omega) = \frac{1}{\pi}\int_0^\infty \left\{ \ln\left(1 + \frac{\omega}{\sqrt{\gamma^2 - k_0^2}}\right) + \frac{\omega}{\sqrt{\gamma^2 - k_0^2}} \right\} f(\gamma)\, d\gamma \tag{4.3}$$

where

$$f(\gamma) = \frac{2}{\pi}\left[\gamma \left\{ J_0^2(\gamma b) + N_0^2(\gamma b) \right\}\right]^{-1} \tag{4.4}$$

Comparison of (3.8) with (4.3) reveals that the latter is somewhat more involved because of the presence of the function $f(\gamma)$ inside the integrand. The corresponding function in the parallel plane waveguide is simply a constant, viz., b.

Equation (4.3) cannot be integrated in a closed form as was possible with (3.8). However, the following observation may be made in regard to the evaluation of R(w). One may verify that $f(\gamma) \approx b + O(1/\gamma^2 b^2)$ large γb and in fact the deviation from this asymptotic value is small even for moderate values of γb.

For the purposes of numerical computations, one may rearrange (4.3) using (3.8) and (3.9) and write it in the form

$$R(\omega) \simeq \frac{b}{\pi} \exp\left\{-\frac{b}{\pi}\sqrt{k_0^2+\omega^2}\,\ln\left(\frac{\omega-\sqrt{k_0^2+\omega^2}}{jk_0}\right)+j\frac{k_0 b}{2}\right\}$$
$$+\int_0^M \left\{\ln\left(1-\frac{\omega}{\sqrt{\gamma^2-k_0^2}}\right)+\frac{\omega}{\sqrt{\gamma^2-k_0^2}}\right\}\left\{f(\gamma)-b\right\}d\gamma \qquad (4.5)$$

where the constant M is chosen on the basis of some accuracy criterion one might set in connection with the infinite integral (4.3).

The above discussion brings out a strong similarity between the cylindrical and parallel plane waveguide problems that may not be apparent without a close look at these problems.

Another problem which belongs to the same category as the two examples discussed above is that of a surface wave launcher shown in Fig. 6b. Also the associated closed region problem is shown in Fig. 6a.

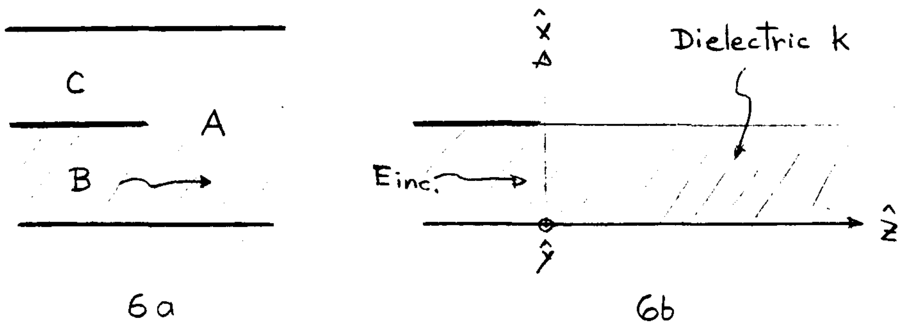

Fig. 6. Surface Wave Launcher and associated closed region problem

The formulation and solution of this problem follows along lines exactly parallel to the ones discussed above. Once more for the closed problem one is able to derive an equation corresponding to (2.19). The quantities β_m's, η_m's and τ_p's are again associated with propagation constants in regions B, C and A, respectively. One is also able to construct an integral representation of $R_c(w)$ and derive $R(w)$ as the limiting case. As before the functions $F_1(\gamma)$ and $F_2(\gamma)$ are the characteristic expressions pertaining to the regions A and C and are easily derivable (see for instance Collin, 1960). An extensive study of this problem has recently been completed by Bates [1965] who has used the Wiener-Hopf technique. He employs a factorization procedure based on the solution of the closed region problem followed by a limiting process. For further details, the interested reader is directed to the reference cited above. Before closing, we shall briefly refer to some related problem which strictly speaking do not conform to the Wiener-Hopf geometry, and to which the present method may be applied with a suitable modification. Two such problems are shown in Fig. 7.

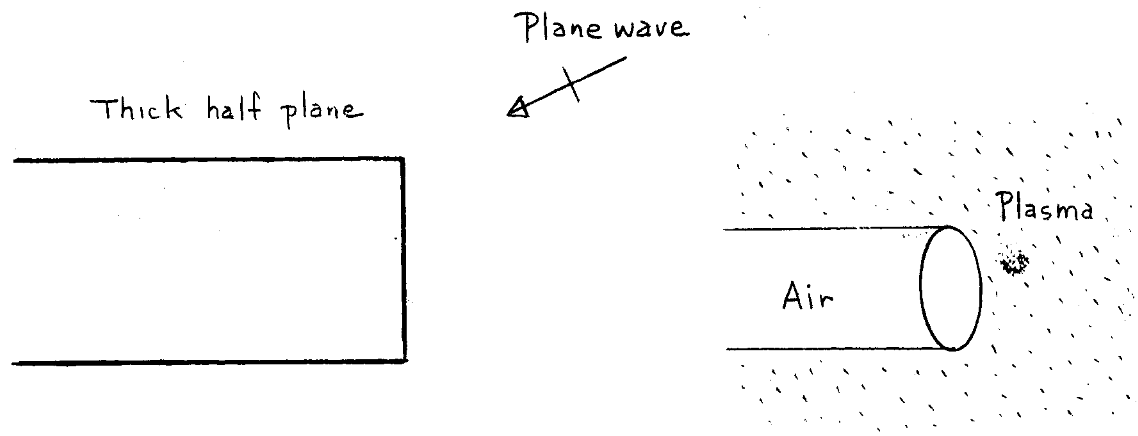

Fig. 7. Geometry of related problems.

First consider the problem of diffraction by a thick half-plane. Although we have not discussed it here, the problem of diffraction of a plane wave by a parallel plane guide may be handled in much the same manner by the method described in Section 2. From the knowledge of this solution one may then develop the solution of the thick half-plane problem by employing a generalized scattering matrix approach. For a discussion of this method refer to Pace and Mittra [op. cit.]. The above reference shows how a rapidly converging series solution may be developed for such problems.

Similar remarks hold for the cylindrical waveguide geometry radiating into a plasma medium. Once again, one might profitably employ the generalized scattering matrix approach to solve this problem. Detailed discussion of these problems is beyond the scope of this paper. However, it is planned to present the analysis of numerous such problems in a future publication.

We might note in passing that the particular method of partitioning the geometry into various regions, e.g. A, B, C and D (see for instance Fig. 1) in the present approach provides a convenient attack of the problems shown in Fig. 7, where one of the basic regions are modified in some manner.

References

1. Bates, C. P., 'A Technique for Solving a Class of Wiener-Hopf Boundary Value Problems' (to be issued).

2. Collin, R. E., 'Field Theory of Guided Waves', McGraw-Hill, 1960 (book).

3. Hurd, R. A. and H. Gruenberg, 'H-Plane Bifurcation of Rectangular Waveguides', Can. J. Phys., 32, 694-701, 1954.

4. Karjala, D. S. and R. Mittra, 'Scattering at the Junction of Two Semi-infinite Parallel Impedance Plane Waveguides', Can. J. Phys., 43, 849-854, 1965.

5. Meixner, J., 'The Behavior of Electromagnetic Fields at Edges', N. Y. Univ., Institute of Math., Sci., Res. Rep. EM-72, 1954.

6. Mittra, R., 'The Finite Range Wiener-Hopf Integral Equation and a Boundary Value Problem in a Waveguide', Trans. IRE, AP-7, Special Supplement, S244-254, 1959.

7. Noble, B., 'Methods Based on the Wiener-Hopf Technique', Pergamon Press, 1958 (book).

8. Pace, J. R., and R. Mittra, 'Generalized Scattering Matrix Analysis of Waveguide Discontinuity Problems', Proc. Quasi-Optics Symp., Vol. XIV, Polytechnic Inst. of Brooklyn, 1964 (book).

9. Whitehead, E. A. N., 'The Theory of Parallel Plate Media for Microwave Lenses', Proc. I. E. E. (London), 98, Pt. III, pp. 133-140, 1951.

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13. ABSTRACT A method is presented for solving a class of radiation and diffraction problems which have conventionally been formulated in terms of the Wiener-Hopf approach. The problem is formulated by representating the fields in terms of a modal expansion. The modal expansion for the field in the open regions is characterized by continuous eigenvalues and for the closed regions the fields are representated in terms of the discrete eigenvectors. Matching of the fields to the boundaries and across the aperture results in an equation which is solved by a technique developed in this article. The technique is an extension of the function-theoretic method necessitated by the field representation in the open regions inherent in the problems. The solutions of an open-ended parallel plate wavequide is used to demonstrate the method. Applications to other geometries are indicated.		

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